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INFINITESIMAL CRITERION FOR FLATNESS OF PROJECTIVE MORPHISM OF SCHEMES

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The generalization of the well-known criterion for flatness of a projective morphism of Noetherian schemes involving Hilbert polynomial, is given for the case of nonreduced base of the morphism.

Bibliography: 4 titles.

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Introduction

We start with some classical notation. Let \mathbb{P}_T^N be relative projective space of dimension N over a scheme T , $\mathcal{O}(1)$ be a line bundle on \mathbb{P}_T^N generated by hyperplane section. It is very ample relative to T . Also if $f : X \rightarrow T$ is a morphism of schemes and $t \in T$ a closed point with residue field $k(t)$ then $X_t := f^{-1}(t)$ is a closed fibre of f over t .

The purpose of the present note is to generalize the following well-known criterion for flatness of a projective morphism of Noetherian schemes [1, ch. III, theorem 9.9]:

Theorem 1. *Let T be an integral Noetherian scheme and $X \subset \mathbb{P}_T^N$ be some closed subscheme. For each closed point $t \in T$ take Hilbert polynomial $P_t \in \mathbb{Q}[m]$ of the fibre X_t . This fibre is considered as closed subscheme in \mathbb{P}_t^N . Then the subscheme X is flat over T if and only if Hilbert polynomial P_t does not depend on the choice of t .*

This theorem is applicable to any projective morphism of schemes $f : X \rightarrow T$ with integral base scheme T , if one reformulate it as follows:

Let projective morphism of Noetherian schemes $f : X \rightarrow T$ with integral scheme T fits into the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{P}_T^N \\ & \searrow f & \downarrow \\ & & T \end{array} \quad (0.1)$$

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with i being closed immersion. It is flat if and only if for an invertible \mathcal{O}_X -sheaf \mathcal{L} which is very ample relative to T and such that $\mathcal{L} = i^*\mathcal{O}(1)$, for every closed point $t \in T$ Hilbert polynomial of the fibre $P_t(m) = \chi(\mathcal{L}^m|_{X_t})$ does not depend on the choice of $t \in T$.

The proof of this theorem as presented in [1] allows to deduce flatness of any coherent \mathcal{O}_X -sheaf \mathcal{F} over an integral scheme T , if Hilbert polynomial $\chi(\mathcal{F} \otimes \mathcal{L}^m|_{X_t})$ of its restriction on the fibre X_t over each point $t \in T$ does not depend on the choice of t .

Cited criterion is not applicable in the case when the scheme T has nonreduced scheme structure (§1). As we will show below (§2), this inconvenience can be removed if Hilbert polynomial is replaced by some other function. In closed points this function coincides with Hilbert polynomial of fibres. We need some notation. If $t \in T$ is a closed point of the scheme T and this point corresponds to a sheaf of maximal ideals $\mathfrak{m}_t \subset \mathcal{O}_X$, then the symbol $t^{(n)}$ stands for n th infinitesimal neighborhood of the point $t \in T$. The n th infinitesimal neighborhood is a subscheme defined by the sheaf of ideals \mathfrak{m}_t^{n+1} in T . In our consideration T is supposed to be Noetherian scheme of finite type over a field, then for each $n \in \mathbb{N}$ the subscheme $t^{(n)}$ is zero-dimensional subscheme of finite length equal to $\text{length } t^{(n)} = \chi(\mathcal{O}_{t^{(n)}})$. It is clear that this is positive integer depending on both t and n . If the point $t = \text{Supp } t^{(n)}$ is known and fixed, we denote the length of n th infinitesimal neighborhood $t^{(n)}$ by the symbol $(n+1)$ (according to the power of maximal ideal corresponding to the subscheme $t^{(n)}$).

We operate in the category of Noetherian schemes over a field k . This field is supposed to be algebraically closed. The hypothesis of algebraic closedness of the base field is essential in those part of argument where we use filtrations (and cofiltrations) of Artinian algebras. Namely, these are proofs of claim 1 and of proposition 4. If A/I is Artinian algebra over an algebraically closed field, then $\text{length } A/I = \dim_k A/I$. Since all vector spaces appearing in this paper are defined over the field k , the lower index in the notation of dimension is omitted.

To deduce that the morphism f is flat, one has to examine preimages $f^{-1}(t^{(n)}) := X \times_T t^{(n)}$ of infinitesimal neighborhoods of reduced points $t \in T$. This provides data on behavior of the morphism f over nonreduced scheme structure of T . Since T is of finite type, the power n to be examined for the given morphism f , is bounded from above (and not greater then maximal index of nilpotent elements in \mathcal{O}_T minus 1). This paper is devoted to the proof of following results (theorem 2 is a particular case of theorem 3, and we prove theorem 3 immediately).

Theorem 2. *Let a projective morphism of Noetherian schemes of finite type $f : X \rightarrow T$ fits into commutative diagram (0.1). It is flat if and only if for an invertible \mathcal{O}_X -sheaf \mathcal{L} very ample relatively T and such that $\mathcal{L} = i^*\mathcal{O}(1)$, for any closed point $t \in T$ the function*

$$\varpi_t^{(n)}(\mathcal{O}_X, m) = \frac{\chi(\mathcal{L}^m|_{f^{-1}(t^{(n)})})}{\chi(\mathcal{O}_{t^{(n)}})}$$

does not depend on the choice of $t \in T$ and of $n \in \mathbb{N}$.

Theorem 3. *Let a projective morphism of Noetherian schemes of finite type $f : X \rightarrow T$ fits into commutative diagram (0.1). The coherent sheaf of \mathcal{O}_X -modules \mathcal{F} is flat with respect to f (i.e. flat as \mathcal{O}_T -module) if and only if for an invertible \mathcal{O}_X -sheaf \mathcal{L} very*

ample relatively T and such that $\mathcal{L} = i^*\mathcal{O}(1)$, for any closed point $t \in T$ the function

$$\varpi_t^{(n)}(\mathcal{F}, m) = \frac{\chi(\mathcal{F} \otimes \mathcal{L}^m|_{f^{-1}(t^{(n)})})}{\chi(\mathcal{O}_{t^{(n)}})}$$

does not depend on the choice of $t \in T$ and of $n \in \mathbb{N}$.

In case when f is finite morphism the function in theorem 2 takes the form

$$\varpi_t^{(n)}(\mathcal{O}_X, m) = \frac{\text{length}(f^{-1}(t^{(n)}))}{\text{length}(t^{(n)})}.$$

If the scheme T is reduced, it is enough to examine only the case $n = 0$. This corresponds to the classical situation $\varpi_t^{(0)}(\mathcal{O}_X, m) = P_t(m)$ (theorem 1).

1 Motivation

Example 1. Consider nonreduced scheme $T = \text{Spec } k[x]/(x^2)$ of length 2 and a morphism $f : X \rightarrow T$ if immersion of (unique) closed point $X = \text{Spec } k$. Since both schemes are supported at a point, we replace examining of a morphism of structure sheaves $f^\# : \mathcal{O}_T \rightarrow f_*\mathcal{O}_X$ by the study of a homomorphism $f^\# : k[x]/(x^2) \rightarrow k$ of corresponding Artinian algebras. It is clear that $f^\#$ is an epimorphism onto the quotient ring over nil-radical: $f^\# : k[x]/(x^2) \rightarrow (k[x]/(x^2))/\text{Nil} = k$. We use the criterion for flatness of a ring homomorphism formulated in [2, ch. 1, proposition 2.1]

Proposition 1. *A homomorphism $f : A \rightarrow B$ is flat if and only if mappings $(a \otimes b \mapsto f(a)b) : I \otimes_A B \rightarrow B$ are injective for all ideals I of A .*

Then it is necessary to test a homomorphism $(x) \otimes_{k[x]/(x^2)} k \rightarrow k$, for injectivity. This homomorphism is induced by the inclusion of the ideal $(x) \hookrightarrow k[x]/(x^2)$. The tensor product $(x) \otimes_{k[x]/(x^2)} k$ is nonzero and is a k -linear span of the element $x \otimes 1$, $x^2 = 0$. The element $x \otimes 1$ is taken to 0 under the mapping to k . Then the ring homomorphism of interest and the corresponding scheme morphism are not flat.

The same result is given by theorem 2. The Hilbert polynomial of the fibre of the morphism f over the unique closed point t of the scheme T equals to $P_t(m) = 1$. The function ϖ if computed for 1st infinitesimal neighborhood of the closed point on the base T (it coincides with the whole of the scheme T), equals to $\varpi_t^{(1)}(m) = 1/2$. This differs from the value $\varpi_t^{(0)}(m) = P_t(m) = 1$.

Example 2. Let $A = k[x]/(x^3)$, and $B = k[x]/(x^2)$ is A -module of interest. It is clear that B is finitely generated (and has one generator) over A . A is local k -algebra with maximal ideal (x) and residue field k . Since B is not free as A -module that B is not flat as A -module. On the other hand, the group $\text{Tor}_1^A(k, B)$ fits into the exact sequence which is induced by tensoring of a triple

$$0 \rightarrow (x^2) \rightarrow A \rightarrow B \rightarrow 0$$

by $\otimes_A k$:

$$0 \rightarrow \text{Tor}_1^A(k, B) \rightarrow (x^2) \otimes_A k \rightarrow A \otimes_A k \rightarrow B \otimes_A k \rightarrow 0 \quad (1.1)$$

This sequence implies that $\mathrm{Tor}_1^A(k, B) = (x^2) \otimes_A k = k$. This also shows that B is not flat as A -module. Computing the function ϖ one has $\varpi_t^{(0)}(m) = \varpi_t^{(1)}(m) = 1$, $\varpi_t^{(2)}(m) = 2/3$.

2 Algebraic version

We will need the following criterion for flatness [3, ch. 1, theorem 7.8].

Proposition 2. *A -module M is flat if and only if $\mathrm{Tor}_1^A(A/I, M) = 0$ for any finitely generated ideal $I \subset A$.*

Convention 1. Let A be local Noetherian k -algebra with residue field $k = \bar{k}$, $I \subset A$ be an ideal such that A/I is Artinian k -algebra of length n , i.e. $\dim A/I = n$. Then the ideal I is said to be of colength n . This fact will be reflected in the notation of the ideal: we write I_n instead of I .

Proposition 3. *Let M be a finitely generated module over the local Noetherian k -algebra A with residue field k . Module M is free if and only if for all $n > 0$ and for all ideals $I_n \subset A$ of colength n*

$$\frac{\dim(M \otimes_A A/I_n)}{n} = \dim(M \otimes_A k). \quad (2.1)$$

Proof. Note that $M/I_n M = M \otimes_A A/I_n$ and $M/I_n M \otimes_{A/I_n} k = M \otimes_A A/I_n \otimes_{A/I_n} k = M \otimes_A k$. Analogously,

$$M/I_n M \otimes_{A/I_n} A/I_{n-1} = M \otimes_A A/I_{n-1} = M/I_{n-1} M.$$

There is an exact triple of A -modules (and of A/I_n -modules)

$$0 \rightarrow \mathfrak{m}_n \rightarrow A/I_n \rightarrow k \rightarrow 0. \quad (2.2)$$

For the maximal ideal $\mathfrak{m}_n \subset A/I_n$ we have $M \otimes_A \mathfrak{m}_n = M \otimes_A A/I_n \otimes_{A/I_n} \mathfrak{m}_n = M/I_n M \otimes_{A/I_n} \mathfrak{m}_n$. Tensoring of (2.2) by $M/I_n M \otimes_{A/I_n}$ yields in the exact sequence

$$0 \rightarrow \mathrm{Tor}_1^{A/I_n}(M/I_n M, k) \rightarrow M/I_n M \otimes_{A/I_n} \mathfrak{m}_n \rightarrow M/I_n M \rightarrow M/I_n M \otimes_{A/I_n} k \rightarrow 0. \quad (2.3)$$

Left exactness is guaranteed by $\mathrm{Tor}_1^{A/I_n}(M/I_n M, A/I_n) = 0$ because any ring is flat over itself.

Claim 1. *The equality (2.1) implies exactness of a sequence*

$$0 \rightarrow M/I_n M \otimes_{A/I_n} \mathfrak{m}_n \rightarrow M/I_n M \rightarrow M/I_n M \otimes_{A/I_n} k \rightarrow 0. \quad (2.4)$$

The proof of this claim will be presented below, when the proof of the proposition will be completed. Claim 1 and exact sequence (2.3) imply that $\mathrm{Tor}_1^{A/I_n}(M/I_n M, k) = 0$. By proposition 2 $M/I_n M$ is flat as A/I_n -module. Now we can consider not all possible ideals of finite colength but powers of maximal ideal $\mathfrak{m} \subset A$ only. Passing to \mathfrak{m} -adic

completions \widehat{A} and \widehat{M} of the ring A and of the module M respectively, we get [3, proof of theorem 22.4(ii)] that \widehat{M} is flat \widehat{A} -module.

By the same theorem [3, theorem 22.4(ii)] we conclude that A -module M is flat.

The proof of the opposite implication is trivial. If the finitely generated module over the local ring is flat then it is free, i.e. $M \cong A^q$. This implies equalities of the form (2.1) for all $n > 0$ and for all ideals $I_n \subset A$. This completes the proof of the proposition. \square

Now we prove claim 1. To organize induction over n consider exact diagrams of the form

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & k & \xlongequal{\quad} & k & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & k & \longrightarrow & A/I_n & \longrightarrow & A/I_{n-1} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & k & \longrightarrow & \mathfrak{m}_n & \longrightarrow & \mathfrak{m}_{n-1} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array} \tag{2.5}$$

Such a diagram of A -modules (and of k -algebras) can be built up for any $n > 0$ and for any ideal $I_n \subset A$ of colength n . For $n = 2$ we have $\mathfrak{m}_n = \mathfrak{m}_2 \cong k$, $\mathfrak{m}_{n-1} = \mathfrak{m}_1 = 0$.

Let (2.1) holds. Then

$$\dim M \otimes_A A/I_n = \dim M \otimes_A A/I_{n-1} + \dim M \otimes_A k.$$

This implies that the triple $0 \rightarrow M \otimes_A k \rightarrow M \otimes_A A/I_n \rightarrow M \otimes_A A/I_{n-1} \rightarrow 0$ is exact.

Remark 1. Exactness of this triple a priori does not imply that

$$\mathrm{Tor}_1^A(M, A/I_{n-1}) = \mathrm{Tor}_1^A(M, A/I_n) = \mathrm{Tor}_1^A(M, k) = 0.$$

This result follows from proposition 3.

Tensoring of the diagram (2.5) by $M \otimes_A$ leads to the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & M \otimes_A k & \xlongequal{\quad} & M \otimes_A k & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & M \otimes_A k & \longrightarrow & M \otimes_A A/I_n & \longrightarrow & M \otimes_A A/I_{n-1} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 & & M \otimes_A k & \longrightarrow & M \otimes_A \mathfrak{m}_n & \longrightarrow & M \otimes_A \mathfrak{m}_{n-1} \longrightarrow 0
 \end{array} \tag{2.6}$$

Let $R := \ker(M \otimes_A \mathfrak{m}_n \rightarrow M \otimes_A \mathfrak{m}_{n-1})$. Then the isomorphism $M \otimes_A k \cong M \otimes_A k$ factors as $M \otimes_A k \twoheadrightarrow R \rightarrow M \otimes_A k$. This implies that $R \cong M \otimes_A k$, and lower horizontal triple in (2.6) is left-exact. Consequently, $\dim M \otimes_A \mathfrak{m}_n = \dim M \otimes_A \mathfrak{m}_{n-1} + \dim M \otimes_A k$. Applying induction over n we have $\dim M \otimes_A \mathfrak{m}_n = (n-1)\dim M \otimes_A k$. This implies that the triple $M \otimes_A \mathfrak{m}_n \rightarrow M \otimes_A A/I_n \rightarrow M \otimes_A k \rightarrow 0$ (which is equivalent to the triple (2.4)) is left-exact. This proves the claim.

Proposition 4. *Equalities (2.1) hold for all $n > 0$ and for all $I_n \subset A$ if and only if the analogous equalities*

$$\dim M \otimes_A A/\mathfrak{m}^n = \dim A/\mathfrak{m}^n \dim M \otimes_A k, \quad (2.7)$$

hold for \mathfrak{m}^n for all $n > 0$.

Proof. Part "only if" is obvious; it rests to prove part "if". Denote by the symbol (n) the length of the quotient algebra A/\mathfrak{m}^n , i.e. $(n) := \dim A/\mathfrak{m}^n$. For descending induction on lengths of quotient algebras we write down exact sequences of the form

$$\begin{aligned} 0 \rightarrow k \rightarrow A/\mathfrak{m}^n \rightarrow A/I_{(n)-1} \rightarrow 0, \\ 0 \rightarrow k \rightarrow A/I_{(n)-1} \rightarrow A/I_{(n)-2} \rightarrow 0, \\ \dots\dots\dots \\ 0 \rightarrow k \rightarrow A/I_{(n')+1} \rightarrow A/\mathfrak{m}^{n'} \rightarrow 0 \end{aligned}$$

for appropriate $n' < n$. Tensoring by $M \otimes_A$ and dimension counting lead to the sequence of inequalities

$$\begin{aligned} \dim M \otimes_A A/\mathfrak{m}^n &\leq \dim M \otimes_A k + \dim M \otimes_A A/I_{(n)-1}, \\ \dim M \otimes_A A/I_{(n)-1} &\leq \dim M \otimes_A k + \dim M \otimes_A A/I_{(n)-2}, \\ \dots\dots\dots \\ \dim A/I_{(n')+1} &\leq \dim M \otimes_A k + \dim A/\mathfrak{m}^{n'}. \end{aligned} \quad (2.8)$$

Continuing descent till $n' = 1$ and applying (2.7) we conclude that inequalities in (2.8) are indeed equalities.

Then for any $I_l, l > 0$ there exist $n > 0$ such that $A/\mathfrak{m}^n \twoheadrightarrow A/I_l$. In this case there is a cofiltration $A/\mathfrak{m}^n \twoheadrightarrow A/I_{(n)-1} \twoheadrightarrow \dots \twoheadrightarrow A/I_l \twoheadrightarrow \dots \twoheadrightarrow k \twoheadrightarrow 0$ of length (n) with kernels isomorphic to k , and such that it contains A/I_l . Counting of dimensions of vector spaces $M \otimes_A A/I_j, j = 1, \dots, l$, and application of equalities (2.8) yield in the required equality $\dim M \otimes_A A/I_l = l \dim M \otimes_A k$. \square

3 Proof for coherent \mathcal{O}_T -module

Since the assertion of the theorem is local in T one can assume that $T = \text{Spec } A$ for local Noetherian k -algebra A . In further text we will use the notation $\mathcal{F}(m) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^m$. The group $H^0(\text{Spec } A, f_* \mathcal{F}(m)) = H^0(X, \mathcal{F}(m))$ carries a structure of finitely generated A -module. It is necessary to prove that this module is flat. For this purpose consider

a finite presentation of the quotient ring A/I_n (it exists because the ideal I_n is finitely generated):

$$A^q \rightarrow A \rightarrow A/I_n \rightarrow 0. \quad (3.1)$$

The quotient ring A/I_n fixes a zero-dimensional subscheme $Z \subset T$ of length n . The presentation (3.1) induces the triple

$$\mathcal{F}(m)^q \rightarrow \mathcal{F}(m) \rightarrow \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^* \mathcal{O}_Z \rightarrow 0.$$

Formation of groups of global sections leads to sequences

$$H^0(X, \mathcal{F}(m))^q \rightarrow H^0(X, \mathcal{F}(m)) \rightarrow H^0(X, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^* \mathcal{O}_Z) \rightarrow 0. \quad (3.2)$$

Right-exactness is achieved here when $m \gg 0$. Tensoring of presentation (3.1) by $H^0(X, \mathcal{F}(m)) \otimes_A$ leads to the right-exact triple

$$H^0(X, \mathcal{F}(m))^q \rightarrow H^0(X, \mathcal{F}(m)) \rightarrow H^0(X, \mathcal{F}(m)) \otimes_A A/I_n \rightarrow 0. \quad (3.3)$$

Comparison of (3.2) and (3.3) yields in the isomorphism

$$H^0(X, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^* \mathcal{O}_Z) = H^0(f^{-1}Z, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^* \mathcal{O}_Z) \cong H^0(X, \mathcal{F}(m)) \otimes_A A/I_n. \quad (3.4)$$

We suppose that

$$\dim H^0(f^{-1}Z, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^* \mathcal{O}_Z) = n \dim H^0(f^{-1}t, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^* k_t), \quad (3.5)$$

where t is the unique closed point of scheme T . By (3.4) we have

$$\begin{aligned} H^0(f^{-1}Z, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^* \mathcal{O}_Z) &\cong H^0(X, \mathcal{F}(m)) \otimes_A A/I_n, \\ H^0(f^{-1}t, \mathcal{F}(m) \otimes_{\mathcal{O}_X} f^* k_t) &\cong H^0(X, \mathcal{F}(m)) \otimes_A k. \end{aligned}$$

Substituting these isomorphisms into (3.5) we conclude that for all $n > 0$ and for all $I_n \subset A$ the following equalities hold:

$$\dim H^0(X, \mathcal{F}(m)) \otimes_A A/I_n = n \dim H^0(X, \mathcal{F}(m)) \otimes_A k,$$

This validates proposition 3 for $H^0(X, \mathcal{F}(m))$. Hence $H^0(X, \mathcal{F}(m))$ is flat A -module.

The proof of flatness of \mathcal{F} as \mathcal{O}_T -module copies the proof of the implication (ii) \Rightarrow (i) in [1, ch. III, proof of theorem 9.9] verbatim. This part of the cited proof remains valid also for nonreduced scheme T . By projectivity of the morphism f we can restrict to the case when f is a structure morphism of some projective bundle $f : \text{Proj } A[x_0 : \dots : x_n] \rightarrow \text{Spec } A$ and consider graded $A[x_0 : \dots : x_n]$ -module $M = \bigoplus_{m \geq m_0} H^0(X, \mathcal{F}(m))$. The integer m_0 is chosen as big as A -modules $H^0(X, \mathcal{F}(m))$ are free for all $m \geq m_0$. Then $\mathcal{F} = \widetilde{M}$, where \sim denotes formation of a coherent sheaf of $\mathcal{O}_{\text{Spec } A}$ -modules which is associated with finitely generated A -module M ("sheafification"). In this case M and $\bigoplus_{m \geq 0} H^0(X, \mathcal{F}(m))$ are equal for all $m \geq m_0$ and hence [1, ch. II, proposition 5.15] $\widetilde{M} = (\bigoplus_{m \geq 0} H^0(X, \mathcal{F}(m)))^\sim$. Since M is free (and, consequently, flat) A -module, then \mathcal{F} is flat over A (and hence over $T = \text{Spec } A$).

The proof of opposite implication repeats the proof of implication (i) \Rightarrow (ii) in [1, ch. III, proof of theorem 9.9] verbatim. Let \mathcal{F} be a flat \mathcal{O}_T -module and we reduce our consideration to the case $X = \text{Proj } A[x_0 : \cdots : x_n]$, $T = \text{Spec } A$, for Noetherian local ring A . Compute $H^i(X, \mathcal{F}(m))$ as Čech cohomology of standard open affine covering \mathfrak{U} of the space X . Namely, $H^i(X, \mathcal{F}(m)) = H^i(C^\bullet(\mathfrak{U}, \mathcal{F}(m)))$. Since the sheaf \mathcal{F} is flat, then for all $i \geq 0$ the term $C^i(\mathfrak{U}, \mathcal{F}(m))$ is flat A -module. If $i > 0$ then for $m \gg 0$ we have $H^i(X, \mathcal{F}(m)) = 0$. Then Čech complex provides a right resolution for A -module $H^0(X, \mathcal{F}(m))$, and the sequence

$$0 \rightarrow H^0(X, \mathcal{F}(m)) \rightarrow C^0(\mathfrak{U}, \mathcal{F}(m)) \rightarrow \cdots \rightarrow C^n(\mathfrak{U}, \mathcal{F}(m)) \rightarrow 0$$

is exact. Since all terms of Čech complex are flat A -modules, then cutting this exact sequence into exact triples we come to flatness of A -module $H^0(X, \mathcal{F}(m))$. Then it is subject of proposition 3, and for all $n > 0$ and for all $I_n \subset A$ following equalities hold:

$$\dim H^0(X, \mathcal{F}(m)) \otimes_A A/I_n = n \dim H^0(X, \mathcal{F}(m)) \otimes_A k.$$

By the isomorphism (3.4) which was proven independently, assertions of theorem 3 are fulfilled.

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References

- [1] R. HARTSHORNE, *Algebraic geometry*, Graduate Texts in Mathematics, **52**, Springer-Verlag, New York – Heidelberg – Berlin, 1977.
- [2] J.S. MILNE, *Étale Cohomology*, Princeton Math. Series, **33**, Princeton Univ. Press, Princeton, New Jersey, 1980.
- [3] H. MATSUMURA, *Commutative ring theory*, Cambridge Univ. Press, Cambridge, 1986.
- [4] M.F. ATIYAH, FRS, and I.G. MACDONALD, *Introduction to commutative algebra*, Addison-Wesley Series in Mathematics, Addison-Wesley Publishing Co., Massachusetts, 1969.